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Decay of Solutions of the Wave Equation Outside Rough Surfaces

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Abstract—In the first part of this paper, we prove the decay of local energy for the solutions of the wave equation in an exterior domain outside a two-dimensional rough surface in $\mathbf{R}^3$ which satisfies an additional geometric condition, "weak star-shaped" condition $\mathcal{WSS}$, implying the absence of trapped rays. Moreover, if the stronger "star-shaped" condition $\mathcal{SS}$ of Morawetz is added, the rate of decay can be bounded by $1/t^2$. We also remark that the result of Ralston remains valid in the "rough surface" case: the existence of trapped rays implies an arbitrarily slow decay of the energy.

If we restrict the analysis to a compact perturbation of a plane, we show that the local energy decays. Moreover, if a geometric condition is added ("star-shaped" condition $\mathcal{SS}$, or "nontrapping" condition $\mathcal{NT}$), we find that the decay is exponential.

Keywords—Scattering, Waves, Asymptotic behaviour, Propagation, Local energy.

1. INTRODUCTION

Scattering of classical waves (or quantum particles) by rough or periodic interfaces is a classical problem actively investigated by physicists, from a theoretical and experimental point of view (see [1–3], and references therein).

Mathematically, this problem has been studied, mainly in the periodic case, by time-independent scattering methods [4,5]. Although the time-dependent aspects of scattering have been, and are presently, the matter of a number of works in the bounded obstacle case (see [6] and references therein), the unbounded obstacle is less studied.

In the present paper, we address the problem of decay in time of local energy for solutions of the wave equation, in a region exterior to an unbounded surface which may be considered as a rough perturbation of a plane.

Wilcox has noted [4] that, for a smooth (say C^∞) periodic surface Γ which admits no Rayleigh-Bloch surface wave, the local energy of the solutions of the exterior Dirichlet problem for the wave equation decays towards zero as time increases.

On the other hand, it is known (see [4]) that if Γ is defined globally as the graph of a smooth function in the case of Dirichlet boundary conditions, such surface waves are actually absent. This geometric condition can be nicely reformulated into a simple angular condition for the exterior normal at each point of Γ. This is a direct extension of the "star-shaped" situation studied by C. Morawetz [7–10], in the compact obstacle case, and we call it "weak star-shaped" condition ($\mathcal{WSS}$). It implies, in particular, that no ray is trapped by Γ [11].

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A classical technique to prove local decay results of energy is the multiplicator method used by Morawetz [7,8] in the bounded obstacle case. This decay depends crucially on geometric properties of the surface ("star-shaped condition"). The natural extension of this condition for a perturbation of a plane P is the global definition of this surface as the graph of a smooth function, which may be seen as a weakened star-shaped condition, where the "star-center" would be removed inside the obstacle, at the infinity in the direction normal to the plane P. So, we are led to estimate the energy in slabs.

Clearly, we expect a very slow decay, because the slab is itself an unbounded region. In fact, we can only show that the local energy decays. We remark that the periodic case of [4] is included in our analysis.

Now if we consider the stronger star-shaped condition of Morawetz, assuming that the surface may be entirely "lit" by a point A interior to the obstacle Ω, with $d(A, \partial\Omega) < \infty$, we show that the decay can be improved, and we obtain $1/t^2$. This improvement is due to the fact that we have an isotropic estimate of energy, controlling also the flow in the tangential directions.

In the opposite case where some rays are trapped by a compact body, Ralston has shown [12] that the decay of the local energy may be arbitrarily slow. His result being of local nature, it is not surprising that it applies also in the case of infinite surfaces.

If now we want to improve the preceding decay, in the spirit of [13], we are faced to some technical difficulties, due to the unboundedness of Γ, and the methods of [9,13], do not seem to be easily transposable. A natural intermediate step is then to study an associated truncated problem.

It is well-known [14] that, in an exterior region with a compact boundary, the Helmholtz equation has a unique solution provided one gives an appropriate boundary condition and a radiation condition. In the case where the boundary extends to infinity, only a few particular cases have been studied, and additional geometric restrictions must be superimposed to insure uniqueness [15–22]. Typically, Alber [23] applies such a geometric condition, first introduced by Eidus, to a periodic grating of bounded obstacles.

Following these remarks, we consider arbitrary compact perturbations of a plane surface, for which one can use a simple adaptation of a result of Kato [24] (already used by Simon [25] in the Schrödinger context), getting a uniqueness result for the associated reduced wave equation. This property, of spectral nature, allows us, using general arguments, to prove a decay result for local energy outside any smooth, compact, perturbation of a plane surface. Moreover, in the special cases of a "star shaped" ($\mathcal{SS}$) perturbation, or a"nontrapping" ($\mathcal{NT}$) perturbation, this decay is shown to be exponential, extending a result of Lax, Morawetz and Phillips (for $\mathcal{SS}$), and Morawetz, Ralston and Strauss (for $\mathcal{NT}$).

The paper is organized as follows: In the first part (Section 2), we give the general decay result. In Section 3, we improve the result for star-shaped surfaces (with a sufficient damping of the roughness at infinity). Section 4 contains a brief description of the result of Ralston, adapted to the periodic framework. In the last section (Section 5), we study the energy decay, first in the general case of a smooth compact perturbation, then, when a "nonconfining" condition is added, leading to an exponential decay.

2. THE DECAY OF LOCAL ENERGY FOR A "WEAK STAR-SHAPED" SURFACE

2.1. Geometry of the Problem and Statement of the Local Decay

We denote the points x of $\mathbb{R}^3$ by (x', x_3), with $x' \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$. Let Ω be a domain in $\mathbb{R}^3$ such that

$$\mathbb{R}^2 \times [h_0, +\infty) \subset \Omega \subset \mathbb{R}^2 \times [h, +\infty), \tag{1}$$

for some $h > h_0 > 0$.

We assume in the following that $\Gamma = \partial\Omega$ is given by the global equation $x_3 = \phi(x')$, where ϕ is a bounded function in $C^\infty(\mathbb{R}^2, \mathbb{R}^+)$, and, for each $x \in \Gamma$, we denote by $\nu = \nu(x)$ the normal exterior to Ω.

We consider the following Dirichlet initial boundary problem for the wave equation in Ω:

$$\begin{aligned} \Box w &= 0, && \text{for } (x,t) \in \Omega \times [0,+\infty), \\ w(x,0) &= f(x), && \text{for } x \in \Omega, \\ \partial_t w(x,0) &= g(x), && \text{for } x \in \Omega, \\ w(x,t) &= 0, && \text{for } (x,t) \in \Gamma \times [0,+\infty). \end{aligned} \tag{2}$$

We suppose that f and g are regular and compactly supported in Ω, and we denote the associated energy by

$$E_0 = \int_\Omega (|g|^2 + |\nabla f|^2)\, dx.$$

The local energy for any solution w of (2), in a given bounded region $D \subset \Omega$, is defined as

$$E(w, D, t) = \int_{D\cap\Omega} (|\partial_t w|^2 + |\nabla w|^2)\, dx.$$

Let us now consider the following geometrical condition (clearly equivalent to the global definition of Γ by ϕ).

CONDITION. (WEAK STAR-SHAPED). *The surface Γ is said to satisfy the weak star-shaped (or $\mathcal{WSS}$) condition if*

$$\forall x \in \Gamma, \qquad \nu(x) \cdot e_3 < 0.$$

Our main result on the energy decay is then found in the following theorem.

THEOREM 1. *If the smooth surface Γ satisfies condition $\mathcal{WSS}$, and if t is large enough, then, for any solution w of (2), and for any bounded region $D \subset \Omega$,*

$$\lim_{t\to+\infty} E(w, D, t) = 0. \tag{3}$$

In [13], Lax, Morawetz and Phillips show that for a compact star-shaped obstacle in $\mathbb{R}^3$, the preceding result may be improved, and that the decay is actually exponential. Their proof relies on the properties of the Lax-Phillips semi-group $Z(t)$ (see [6]). In our case, the obstacle is unbounded, and their technique is not directly transposable.

In [9], Morawetz uses a more direct argument to prove exponential decay, using an iterative splitting of the solution into a "free" part, well controlled, and a remainder with a reduced energy. However this procedure is connected with the fact that any signal will leave the surface of the obstacle after a finite time, and this is no longer valid for an unbounded obstacle.

As a matter of fact, we remark that in the trivial case of the flat surface, the decay is really exponential, due to a trivial extension of the Huygens principle. So, at least in the periodic case and if the parameter h controlling the height of the surface Γ is small enough, it is reasonable to make the following guess, to be investigated in a future work.

CONJECTURE 1. *If the periodic surface Γ satisfies condition $\mathcal{WSS}$, then, for a solution w of the problem (2), their exists a positive number h_0 such that, for $0 \le h \le h_0$,*

$$E(w, D, t) \le C\, e^{-\lambda t}\, E(w, \Omega, 0), \tag{4}$$

where the constants C and λ are positive numbers depending on the geometry of Γ and the supports of the initial data.

2.2. Proof of Theorem 1

In this section, we prove that local energy decays toward zero as time increases, for a solution of the problem:

$$\begin{aligned} \Box w &= 0, && \text{for } (x,t) \in \Omega \times [0,+\infty), \\ w(x,0) &= f(x), && \text{for } x \in \Omega, \\ \partial_t w(x,0) &= g(x), && \text{for } x \in \Omega, \\ w(x,t) &= 0, && \text{for } (x,t) \in \Gamma \times [0,+\infty). \end{aligned} \tag{5}$$

We consider this problem in the following abstract setting:

$$\begin{aligned} \frac{d\mathcal{U}}{dt} &= \mathcal{A}\mathcal{U}, \qquad \text{for } t > 0, \\ \mathcal{U}(0) &= \mathcal{U}_0, \end{aligned} \tag{6}$$

where $\mathcal{U}(t) = \begin{pmatrix} w \\ w_t \end{pmatrix}$, $\mathcal{U}_0 = \begin{pmatrix} f \\ g \end{pmatrix}$, and the operator: $\mathcal{A} = \begin{pmatrix} 0 & I \\ -\Delta_D & 0 \end{pmatrix}$, has domain $D(\mathcal{A}) = D(-\Delta_D) \times H^1(\Omega)$, Δ_D being the Laplacian with Dirichlet boundary condition on $\Gamma = \partial\Omega$. We shall use the notation $A = -\Delta_D$.

Then, the solution of (5) is

$$w(t) = \cos(A^{1/2}t)f + A^{-1/2} \cdot \sin(A^{1/2}t)g. \tag{7}$$

Let us show first the following lemma.

LEMMA 1. *Suppose that u is a L^2 solution of the Dirichlet boundary problem for the reduced wave equation in Ω:*

$$\begin{aligned} \Delta u + \omega^2 u &= 0, && \text{for } x \in \Omega,\ \omega \in \mathbb{R}, \\ u &= 0, && \text{for } x \in \Gamma. \end{aligned} \tag{8}$$

Let us denote by Ω_R the slab $\Omega \cap \{x_3 \le R\}$.

Then the following inequality holds, for each positive R,

$$\int_{\Omega_R} (|\nabla u|^2 + \omega^2 u^2)\ dx < +\infty. \tag{9}$$

PROOF. First we observe that the following integral on the plane $P_R = \{x_3 = R\}$

$$\int_{P_R} u^2(x)\, dx', \tag{10}$$

cannot increase if R is large enough, say $R \ge R_0$.

Thus, there exists a sequence $R_n \to +\infty$ such that

$$\frac{\partial}{\partial x_3} \int_{P_{R_n}} u^2(x)\, dx' \le 0, \tag{11}$$

then

$$\int_{P_{R_n}} \frac{\partial u}{\partial x_3} u\ dx' \le 0. \tag{12}$$

Using Green formula in Ω_{R_n},

$$\int_{\Omega_{R_n}} u(\Delta u + \omega^2 u)\ dx = \int_{\Omega_{R_n}} \omega^2 u^2\ dx + \int_{P_{R_n}} \frac{\partial u}{\partial x_3} u\ dx' + \int_{\Omega_{R_n}} |\nabla u|^2\ dx. \tag{13}$$

So:

$$\int_{\Omega_{R_n}} (\omega^2 u^2 - |\nabla u|^2)\ dx + \int_{P_{R_n}} \frac{\partial u}{\partial x_3} u\ dx' = 0. \tag{14}$$

Then, using (12),

$$\int_{\Omega_{R_n}} |\nabla u|^2\ dx \le \omega^2 \int_{\Omega_{R_n}} u^2\ dx < +\infty. \tag{15}$$

Taking the limit, we have finally (9). ∎

Now we can show the following lemma.

LEMMA 2. *The operator $A = -\Delta_D$ has no positive eigenvalue.*

PROOF. Let u be a nonzero solution of (8).

Using $\partial_3 u = \frac{\partial u}{\partial x_3}$ as a multiplicator, we have

$$\int_{\Omega_R} \partial_3 u(\Delta u + \omega^2 u)\, dx = 0. \tag{16}$$

Putting in the divergence form, we have

$$\int_{\Omega_R} \left(\nabla \cdot (\partial_3 u . \nabla u) - \frac{1}{2}\partial_3 |\nabla u|^2 + \frac{1}{2}\omega^2 \partial_3 u^2\right)\ dx = 0. \tag{17}$$

Using Gauss formula, we find

$$\int_{\partial\Omega_R} \left(\partial_3 u \cdot (\nabla u \cdot \nu) - \frac{1}{2}|\nabla u|^2 (\nu \cdot e_3) + \frac{1}{2}\omega^2 u^2 (\nu \cdot e_3)\right) d\Gamma = 0. \tag{18}$$

The contribution of Γ is, using the boundary condition

$$\int_\Gamma \left(\partial_3 u \cdot \partial_\nu u) - \frac{1}{2}(\partial_\nu u)^2 (\nu \cdot e_3)\right) d\Gamma = \int_\Gamma \frac{1}{2}(\partial_\nu u)^2 (\nu \cdot e_3)\, d\Gamma. \tag{19}$$

The contribution of the plane P_R is

$$\int_{P_R} \left((\partial_3 u)^2 - \frac{1}{2}|\nabla u|^2 + \frac{1}{2}\omega^2 (u^2)\right)\ dx. \tag{20}$$

Collecting (19) and (20), we find, denoting by ∇' the gradient in the x' variables,

$$\int_{P_R} \left((\partial_3 u)^2 - |\nabla' u|^2 + \omega^2 u^2\right)\ dx' = -\int_\Gamma (\partial_\nu u)^2 (\nu \cdot e_3)\, d\Gamma. \tag{21}$$

Now, using the $\mathcal{WSS}$ condition, we have for a positive η, the inequality

$$-\int_\Gamma (\partial_\nu u)^2 (\nu \cdot e_3)\, d\Gamma \ge \eta > 0. \tag{22}$$

Then, in (21),

$$\int_{P_R} \left((\partial_3 u)^2 - |\nabla' u|^2 + \omega^2 u^2\right)\ dx' \ge \eta > 0. \tag{23}$$

So, with a trivial majorization,

$$\int_{P_R} \left(|\nabla u|^2 + \omega^2 u^2\right)\ dx' \ge \eta > 0. \tag{24}$$

Integrating in R, we have then

$$\int_{\Omega_R} \left(|\nabla u|^2 + \omega^2 u^2\right) dx' \geq \eta R. \tag{25}$$

This quantity tends to infinity with R, but, due to Lemma 1, this implies that u cannot be L^2, which proves the lemma. ∎

We have now the following spectral result.

PROPOSITION 1. *The operator A enjoys the following spectral properties:*

(1) *Its spectrum is absolutely continuous.*
(2) *If $\mathcal{E}(A^{1/2},\lambda)$ is the spectral family of $A^{1/2}$, the functions*

$$\int_{-\infty}^{+\infty} e^{\pm i\lambda t} d\left(\mathcal{E}(A^{1/2},\lambda)u, v\right), \tag{26}$$

tend to zero as $|t| \to \infty$, in $L^2(\Omega)$.

PROOF.

(1) It is sufficient to show that, if $\mathcal{E}(A,\lambda)$ is the spectral family associated to A, the function $\lambda \to (\mathcal{E}(A,\lambda)u,u)$ is continuous, for a dense set of u (see [26]).

By Lemma 2, we know that A has no eigenvalue. So $\mathcal{E}(A,\lambda)$ is continuous, and, using the Stone formula [27], we get

$$(\mathcal{E}_{[0,a]}u, v) = \lim_{\epsilon \to 0} \frac{1}{2i\pi} \int_0^a \left(\left[(A-\lambda-i\epsilon)^{-1} - (A-\lambda+i\epsilon)^{-1}\right] u, v\right) d\lambda. \tag{27}$$

Now, by a staightforward extension of the bounded obstacle situation, we see that the limit is a continuous function with respect to λ, for $\epsilon > 0$ (or $\epsilon < 0$). Then we commute the limit and the integral in (27). This shows that the function $\lambda \to (\mathcal{E}(A,\lambda)u,u)$ is continuous, when u varies in the set of L^2 functions with compact support, which is dense set in L^2.

(2) By analogous arguments, the function $\lambda \to \mathcal{E}(A^{1/2},\lambda)$ is absolutely continuous. So there exists $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ such that: $\left(\mathcal{E}(A^{1/2},\lambda)u,v\right) = \int_0^\lambda f(s)\, ds$.

By the properties of spectral measures [27]: $\lim_{\lambda\to+\infty}\left(\mathcal{E}(A^{1/2},\lambda)u,v\right) = (u,v)$, and $\left(\mathcal{E}(A^{1/2},\lambda)u,v\right) = 0$, if $\lambda < 0$.

As a consequence:

$$\int_{-\infty}^{+\infty} f(s)\, ds = \int_0^{+\infty} f(s)\, ds < \infty.$$

So, $f \in L^1(\mathbb{R})$, and (26) is a simple consequence of the Riemann-Lebesgue Lemma.

∎

Now, we can prove the following decay result.

PROPOSITION 2. *For any initial data $(f,g) \in H^1 \times L^2$, and any ball B_R with center zero and radius R, the function w defined by (7) satisfies:*

$$\lim_{t\to+\infty} w(t) = 0,$$

strongly in $L^2(B_R \cap \Omega)$.

PROOF. By a commutation argument, we can write (7) as:

$$w(t) = \cos(A^{1/2}t)f + \sin(A^{1/2}t)A^{-1/2}g,$$

with $f \in H^1$, and $g \in L^2$.

Then $f' \equiv A^{-1/2}g \in L^2$, and if we denote by $(.,.)$ the scalar product in L^2, we have, for any test function $\theta \in L^2$,

$$(w,\theta) = \left(\left[\frac{1}{2}\left(e^{iA^{1/2}t} + e^{-iA^{1/2}t}\right) f - \frac{1}{2i}\left(e^{iA^{1/2}t} - e^{-iA^{1/2}t}\right)\right] f', \theta\right).$$

The four terms in the right hand side, have the same form: $\left(e^{iA^{1/2}t}h, \theta\right)$, then, by the spectral theorem, we obtain for each of them

$$\left(e^{iA^{1/2}t}h, \theta\right) = \int_{-\infty}^{+\infty} d\left(\mathcal{E}(A^{1/2}, \lambda)h, \theta\right),$$

and, after Proposition 1, this quantity tends to zero.

Then, we find that

$$\forall \theta \in L^2: \qquad \lim_{t \to +\infty} (w, \theta) = 0. \tag{28}$$

In the same way, if $f \in H^1 = D(A^{1/2})$,

$$A^{1/2}w = \cos(A^{1/2}t)(A^{1/2}f) + \sin(A^{1/2}t)g$$

then, by the preceding arguments, we get

$$\forall \theta \in H^1: \qquad \lim_{t \to +\infty} \left(A^{1/2}w, A^{1/2}\theta\right) = 0. \tag{29}$$

By norm equivalency in $H^1(\mathbb{R}^3)$, we get

$$||w||^2_{H^1(\mathbb{R}^3)} \le C(||w||^2_{L^2(\mathbb{R}^3)} + a(w,w)),$$

where a is the bilinear form associated to A: $a(w,w) = \left(A^{1/2}w, A^{1/2}w\right)$.

So, (28) and (29) are equivalent to

$$\lim_{t \to +\infty} w(t) = 0, \text{ weakly in } H^1(\mathbb{R}^3),$$

then it also holds weakly in $H^1(B_R)$. So the proof is achieved, using the Rellich compactness theorem. ∎

As we have supposed that the boundary is regular, we can improve the result as follows.

THEOREM 2. *For any initial data $(f,g) \in C_0^\infty$, with support in $\Omega \cap B_R$, the local energy $E(w,t,\Omega \cap B_R)$ of the solution of (5) satisfies*

$$\lim_{t \to +\infty} E(w,t,\Omega \cap B_R) = 0. \tag{30}$$

PROOF. By (7), we have

$$w'(t) = -\sin(A^{1/2}t)A^{1/2}f + \cos(A^{1/2}t)g$$

and

$$A^{1/2}w'(t) = -\sin(A^{1/2}t)(Af) + \cos(A^{1/2}t)(A^{1/2}g),$$

where all the arguments are in L^2.

This shows, as in the proof of Theorem 4, that

$$\lim_{t \to +\infty} ||w'||_{L^2(B_R)} = 0. \tag{31}$$

But we have also

$$Aw(t) = \cos(A^{1/2}t)(Af) + \sin(A^{1/2}t)(A^{1/2}g),$$

with arguments in L^2.

This implies

$$\lim_{t \to +\infty} Aw(t) = 0, \text{ weakly in } L^2(\mathbb{R}^3), \tag{32}$$

and we have seen that

$$\lim_{t \to +\infty} w(t) = 0, \text{ weakly in } L^2(\mathbb{R}^3). \tag{33}$$

By standard elliptic regularity, we have the estimate

$$||w||^2_{H^2(B_R)} \le C(||Aw||^2_{L^2} + ||w||^2_{L^2}).$$

By (32) and (33), we obtain

$$\lim_{t \to +\infty} w(t)|_{B_R} = 0, \text{ weakly in } H^2(\mathbb{R}^3).$$

By the Rellich Theorem,

$$\lim_{t \to +\infty} w(t)|_{B_R} = 0, \text{ strongly in } L^2(B_R)$$

and

$$\lim_{t \to +\infty} \nabla w(t)|_{B_R} = 0, \text{ strongly in } L^2(B_R),$$

which, together with (31), implies (30). ∎

3. THE DECAY OF LOCAL ENERGY FOR A "STAR-SHAPED" SURFACE

We are going to show that if we put more constraint on the geometry, we can give an estimate for the decay of the local energy.

More precisely, we consider the following slight modification of the standard "star-shaped" condition of Morawetz (see [7]).

CONDITION 2 (STAR-SHAPED). *The surface Γ is said to satisfy the* "**star-shaped**" *(or SS) condition if the origin of the coordinates can be chosen in such a way that*

$$\forall \mathbf{x} \in \Gamma, \qquad \mathbf{n} \cdot \mathbf{x} < 0,$$

where $\mathbf{n}$ is the normal at $\mathbf{x}$ pointing out of Ω.

In other words, each point x on Γ can be illuminated by a ray issued from the origin.

This stronger condition implies also that the oscillations of Γ in the slab $h_0 \le x_3 \le h$ are more and more damped as $|x|$ tends to infinity.

In this case, we have the following result.

THEOREM 3. *If the smooth surface Γ satisfies condition SS, then, if t is large enough, for any solution w of* (2), *and for any bounded region $D \subset \Omega$:*

$$E(w, D, t) \le \frac{C}{t^2} \cdot E(w, \Omega, 0), \tag{34}$$

where the positive constant C depends only on the geometry of Γ and the supports of the initial data.

PROOF. The idea is to use an isotropic multiplicator analogous to that introduced by Morawetz in [7], which allows a control on the "tangential flow" of energy.

First, one can verify by a simple computation, the following differential equality, if w is the solution of (2)

$$2Nw \cdot \Box w = 0 = \nabla \cdot \mathbf{F} + G_t, \tag{35}$$

where

(1) N is the first-order differential operator given by

$$Nw = (r^2 + t^2)w_t + 2rtw_r + 2tw. \tag{36}$$

(2) $\mathbf{F}$ is the vector

$$\mathbf{F} = 2((t^2 + r^2)w_t + 2rtw_r + 2tw)\nabla w - 2t\mathbf{x}(|\nabla w|^2 - w_t^2), \tag{37}$$

(3) G is the scalar

$$G = (t^2 + r^2)(|\nabla w|^2 + w_t^2) + 4trw_t w_r + 4tww_t - 2w^2. \tag{38}$$

Now, we integrate (35) in the domain $\Omega \times [0, T]$:

$$\int_0^T dt \int_\Omega \nabla \cdot \mathbf{F}\, dx = \int_0^T dt \int_\Omega G_t\, dx. \tag{39}$$

By the divergence theorem, the integral in the left-hand side gives

$$\int_0^T dt \int_\Gamma \mathbf{F} \cdot \mathbf{n}\, d\Gamma. \tag{40}$$

On Γ, the Dirichlet boundary condition gives: $w = w_t = 0$, then putting (36) into the surface integral gives

$$\int_0^T 2 \int_\Gamma \left[2rtw_r \nabla w - 2t\mathbf{x}|\nabla w|^2\right] \cdot \mathbf{n}\, d\Gamma\, dt, \tag{41}$$

which is

$$\int_0^T 2 \int_\Gamma t\left[2(\mathbf{x} \cdot \nabla w)(\mathbf{n} \cdot \nabla w) - (\mathbf{x} \cdot \mathbf{n})|\nabla w|^2\right]\, d\Gamma\, dt. \tag{42}$$

But on $\Gamma, \nabla w = \frac{\partial w}{\partial n} n$, then (42) becomes

$$\int_0^T 2 \int_\Gamma t\left[2(\mathbf{x} \cdot \mathbf{n})\left(\frac{\partial w}{\partial n}\right)^2 - (\mathbf{x} \cdot \mathbf{n})\left(\frac{\partial w}{\partial n}\right)^2\right]\, d\Gamma\, dt. \tag{43}$$

By the $\mathcal{SS}$ condition, this quantity is nonpositive, and we obtain finally the inequality

$$\int_\Omega \left[(r^2 + T^2)(|\nabla w|^2 + w_t^2) + 4rTw_r w_t + 4Tww_t - 2w^2\right]\, dx \leq \int_\Omega r^2\left(|\nabla w|^2 + w_t^2\right) - 2w^2|_{t=0}\, dx, \tag{44}$$

which can be minorized as

$$\int_\Omega \left[(r^2 + T^2)(|\nabla w|^2 + w_t^2) - 4rT|w_r w_t| - 4T|ww_t| - 2w^2\right]\, dx \leq \int_\Omega r^2\left(|\nabla w|^2 + w_t^2\right) - 2w^2|_{t=0}\, dx. \tag{45}$$

If we suppose that the support of the initial data is inside the ball $B(0,R)$, the right hand side of (45) is less than R^2E_0.

Now, we have to absorb the term $-2w^2$, which has the "wrong" sign.

LEMMA 3. *For each solution w of (2), we have the estimate*

$$\int_\Omega (r^2|\nabla w|^2 - 2w^2)\,dx \geq 0. \tag{46}$$

PROOF. For arbitrary a and $\mathbf{b}$, we have

$$\int_\Omega (a|\nabla w| + \mathbf{b}w)^2\,dx \geq 0. \tag{47}$$

Developing the square, we find

$$\int_\Omega (a^2|\nabla w|^2 + a\mathbf{b}\nabla(w^2) + \mathbf{b}^2w^2)\,dx \geq 0. \tag{48}$$

Now, using the identity,

$$a\mathbf{b}\nabla(w^2) = \nabla\cdot(a\mathbf{b}w^2) - w^2\nabla\cdot(a\mathbf{b}), \tag{49}$$

and taking into account the finite speed of propagation for any solution of (2), we find

$$\int_\Omega a\mathbf{b}\nabla(w^2)\,dx = -\int_\Omega w^2\nabla\cdot(a\mathbf{b})\,dx. \tag{50}$$

So, putting in (48),

$$\int_\Omega \left[(a^2|\nabla w|^2 - w^2(\nabla\cdot(a\mathbf{b}) - \mathbf{b}^2)\right]\,dx \geq 0. \tag{51}$$

Identifying with (46) gives

$$a = r,$$

and

$$\nabla\cdot(r\mathbf{b}) - \mathbf{b}^2 = 2.$$

The simplest choice for $\mathbf{b}$ is $\lambda\frac{\mathbf{x}}{r}$, which gives

$$\nabla\cdot(r\mathbf{b}) = 3\lambda. \tag{52}$$

Then, we have the equation $\lambda^2 - 3\lambda + 2 = 0$, which gives, for any of its solutions $\lambda = 1$ or 2, the estimate (46). ∎

Using Lemma 2 in (45), we are left with the inequality

$$\int_\Omega \left[(r^2+T^2)w_t^2 + T^2|\nabla w|^2 - 4rT|w_rw_t| - 4T|ww_t|\right]\,dx \leq R^2E_0. \tag{53}$$

So, using the inequalities,

$$-4rt|w_rw_t| \geq -2rt(w_r^2 + w_t^2),$$

and

$$-4t|ww_t| \geq -2t(w^2 + w_t^2),$$

we find

$$\int_\Omega \left[(r^2+T^2)w_t^2 + T^2|\nabla w|^2 - 2rT(w_r^2+w_t^2) - 2T(w^2+w_t^2)\right]\,dx \leq R^2E_0 \tag{54}$$

or

$$\int_\Omega \left[(r^2 + T^2 - 2rT - 2T)w_t^2 + T^2|\nabla w|^2 - 2rTw_r^2 - 2Tw^2\right] dx \le R^2 E_0. \tag{55}$$

Now, using Lemma 3 once more,

$$\int_\Omega \left[(r-T)^2 - 2T\right] w_t^2 + T^2|\nabla w|^2 - 2rTw_r^2 - T|\nabla w|^2 \, dx \le R^2 E_0 \tag{56}$$

and

$$\int_\Omega \left[(T-r)^2 - 2T\right] w_t^2 + \left[T^2 - 2rT - T\right] |\nabla w|^2] \, dx \le R^2 E_0. \tag{57}$$

In the region $r \le \epsilon T$,

$$\int_{r \le \epsilon T} \left[T^2(1-\epsilon)^2 - T\right] w_t^2 + \left[T^2(1-2\epsilon^2) - T\right] |\nabla w|^2\right] dx \le R^2 E_0. \tag{58}$$

So,

$$\left(T^2(1-2\epsilon^2) - T\right) \int_{r \le \epsilon T} \left[w_t^2 + |\nabla w|^2\right] dx \le R^2 E_0. \tag{59}$$

Then, for T large enough, we can take $\epsilon = R/T$, and for any bounded region $D \subset \Omega$, we get the required decay

$$\int_D \left[w_t^2 + |\nabla w|^2\right] dx \le \frac{C}{T^2} \cdot E_0, \tag{60}$$

where C is a positive constant.

REMARKS.

(1) The $1/t^2$-decay given by Theorem 3 can be related to the conformal invariance of the equations. This invariance has been used by Glassey and Strauss [28] for the Yang-Mills equations, and can also be applied to the Maxwell system, which bears the same global symmetry [29].

(2) One can say that this decay is "geometrically optimal" in the following sense: the other invariants listed in [28] (see also [30]) do not give a result better than quadratic.

4. SLOW DECAY FOR TRAPPING SURFACES

Let Γ be a surface which is a plane with a finite "bump" inside (see Figure 1).

Following Ralston [12], we define an admissible path in the bounded region:

$$\omega_R = \{x \in \Omega : \ |x| < R\}$$

as a finite "broken line" $P_0, P_1 \dots P_N, P_{N+1}$ such that for $1 \le j \le N$: $P_j \in \gamma_R \equiv \Gamma \cap \omega_R$, while P_0 and P_{N+1} are two points in ω_R, and the P_j are built by successive "elastic reflections" on γ_R (Figure 1).

Let $\delta(\omega_R)$ be the maximum of the lengths of admissible paths in ω_R. Then, we have the following theorem.

THEOREM 4. *Suppose that $\delta(\omega_R) = \infty$. Then, for any $\epsilon > 0$, and any $T > 0$, there exists initial data f and g in $C_0^\infty(\omega_R)$ with energy*

$$E_0 = \int_{\omega_R} (|g|^2 + |\nabla f|^2) \, dx,$$

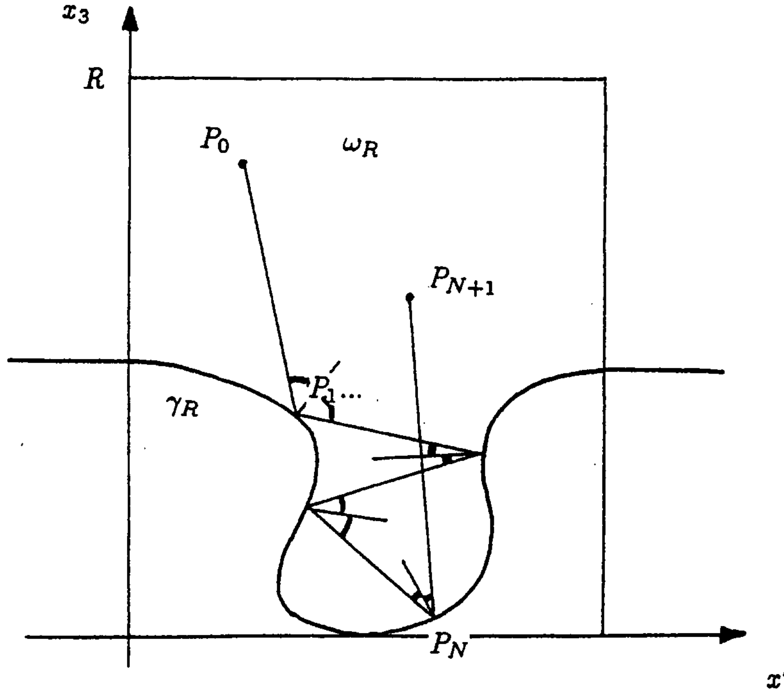

Figure 1. The periodic trap.

such that the solution of

$$\begin{aligned} \Box w &= 0, && \text{for } (x,t) \in \Omega \times [0,+\infty), \\ w(x,0) &= f(x), && \text{for } x \in \Omega, \\ \partial_t w(x,0) &= g(x), && \text{for } x \in \Omega, \\ w(x,t) &= 0, && \text{for } (x,t) \in \Gamma \times [0,+\infty), \end{aligned} \tag{61}$$

satisfies

$$\int_{\omega_R} (|\partial_t w|^2 + |\nabla w|^2)\, dx > (1-\epsilon) E_0.$$

The proof is an adaptation of [12].

5. THE CASE OF A COMPACT PERTURBATION OF A PLANE SURFACE

The new feature is that, as the perturbation of the plane boundary is compact, we can relax the $\mathcal{WSS}$ condition.

5.1. A Uniqueness Result for the Reduced Wave Equation

We extend the result of Lemma 2.

5.1.1. Geometry of the problem and statement of the result

We consider, for any positive number R, a surface Γ_R, which is an imbedded, two-dimensional, C^∞, orientable submanifold of $\mathbb{R}^3$, and we assume that Γ_R is identical to the plane $\Pi_0 = \{(x', x_3)\colon\ x_3 = 0\}$ outside a compact region.

Then Γ_R is an arbitrary "rough" perturbation of Π_0, for which we do not suppose the "weak star-shaped" condition $\mathcal{WSS}$: the convolutions of Γ_R may trap local energy (as seen in Section 4), but we shall see that they do not trap eigenmodes.

We denote in the same way by Ω_R, the associated rough perturbation of the half-space $\mathbb{R}^3$, which is the exterior region.

Then we consider the following Dirichlet boundary problem for the reduced wave equation in Ω_R:

$$\begin{aligned} \Delta u + k^2 u = 0 \qquad & \text{for } x \in \Omega_R,\ k \in \mathbb{R} \\ u = 0 \qquad & \text{for } x \in \Gamma_R, \end{aligned} \tag{62}$$

together with a radiation condition adapted to an infinite boundary [15]:

$$\lim_{\rho \to \infty} \int_{S_\rho \cap \Omega_R} \left| \frac{\partial u}{\partial r} - iku \right|^2 dS = 0, \tag{63}$$

where S_ρ is the 2-sphere of radius ρ.

Our uniqueness result is:

PROPOSITION 3. *For arbitrarily large R, any solution u of* (62) *is trivial.*

Unfortunately, the method we use does not tell us if the preceding result is valid uniformly in R, that is to say if we can remove the cut-off, but it seems unlikely that some pathology might appear at the limit $R \to +\infty$, strong enough to trap an eigenmode.

5.1.2. Proof of Proposition 3

Relying on the analysis of Kato [24], we consider a given solution of (62), and we separate the radial variable $r = |x|$, from the angular variable ω living on the two-sphere S_2.

With this splitting, u may be considered as a function from $\mathbb{R}^+$ into $L^2(S_2)$. Then, (62) can be written as a second order ordinary differential equation

$$u''(r) + \frac{2}{r}u'(r) - \frac{1}{r^2}\Delta_S u(r) + k^2 u(r) = 0, \tag{64}$$

where $'$ denotes the Frechet derivative of $r \to u(r) \in L^2(S_2)$, and Δ_S is the Laplace-Beltrami operator on S_2, given, in spherical coordinates (θ, ϕ) by

$$\Delta_S = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}.$$

If we denote by $(\ ,\ )$ the scalar product in $L^2(S_2)$, and by $||.||$ the associated norm, we have, for any $f \in L^2(S_2)$

$$(\Delta_S f, f) = \int \left[\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2 \right] \sin\theta \, d\theta \, d\phi. \tag{65}$$

Let us consider the function defined by $v_m = r^{m+1}u$, for $m > 0$, with $v_0 = ru \equiv v$. It satisfies the equation

$$v_m''(r) - \frac{2m}{r}v_m'(r) + \frac{1}{r^2}\left[m(m+1) - \Delta_S\right] v_m(r) + k^2 v_m(r) = 0. \tag{66}$$

Then we introduce the following function

$$F(m,t,r) = ||v_m'||^2 + \left(k^2 - \frac{2kt}{r} + \frac{m(m+1)}{r^2}\right)||v_m||^2 - \frac{1}{r^2}(\Delta_S v_m, v_m), \tag{67}$$

where m and t are two constants.

Then Kato shows that F enjoys the following properties (see [24], Lemmas 1–3):

PROPOSITION 4. (KATO). *Let t_0 be a constant such that $0 < t_0 < kR_0$.*

There exists $m_1 \geq 0$ and $R_1 \geq R_0$ such that:

(1) $\frac{d}{dr}\left(r^2 F(m, t_0, r)\right) \geq 0$, *for $m \geq m_1$, and $r \geq R_0$,*

(2) $F(m_0, t_0, r) > 0$, *for $r \geq R_1$,*

(3) *If we assume that $||v||$ is not monotone increasing in any semi-infinite interval of the form $[\rho, +\infty)$, then there exist arbitrary large values of r for which $F(0,0,r) > 0$.*

Using this result, we have the following theorem.

LEMMA 4. *We have the following asymptotic behavior for v:*

$$\underline{lim}_{r\to\infty}\left(||v'||^2+||v||^2\right)>0.$$

PROOF. First we notice that, if $||v||$ is monotone increasing, the result is trivial. If it is not the case, part 3) of Proposition 2 applies. Let us consider the expression (67), with $m=t=0$.

After a short computation taking into account the symmetry of Δ_S, we have

$$\frac{d}{dr}F(0,0,r)=2\Re e\left(v',v''+k^2v-\frac{1}{r^2}\Delta_S v\right)+\frac{1}{r^2}\left(\Delta_S v,v\right). \tag{68}$$

But after (66), we have

$$v''-\frac{1}{r^2}\Delta_S v+k^2v=0.$$

Then, putting in (68) gives

$$\frac{d}{dr}F(0,0,r)=\frac{1}{r^2}\left(\Delta_S v,v\right)\geq 0. \tag{69}$$

We have also

$$F(0,0,r)=||v'||^2+k^2||v||^2-\frac{1}{r^2}\left(\Delta_S v,v\right). \tag{70}$$

Then,

$$F(0,0,r)\leq ||v'||^2+k^2||v||^2. \tag{71}$$

The proof of the lemma is then completed if we consider equation (71), together with part 3 of Proposition 2. ∎

Now, we need a regularity result for a solution of (62) shown as follows.

LEMMA 5. *Any solution $u\in L^2(\Omega_R)$ of (62) satisfies $\nabla u\in L^2(\Omega_R)$.*

PROOF. After multiplying (62) by $\overline{u}$, we integrate over the region $B_r=\{x\in\Omega:\ |x|<r,\}$, with $r>R$

$$0=\int_{B_r}\overline{u}(\Delta u+k^2u)\,dx=k^2\int_{B_r}|u|^2\,dx-\int_{B_r}|\nabla u|^2\,dx+\int_{\partial B_r}\overline{u}\,\frac{\partial u}{\partial n}\,ds.$$

The surface integral splits into two parts: one contribution on Γ, which is zero, due to the boundary condition, the other on the hemisphere.

Then we obtain

$$\int_{B_r}|\nabla u|^2\,dx=\ k^2\int_{B_r}|u|^2\,dx+\Re e\int_{\partial B_r}\overline{u}\,\frac{\partial u}{\partial r}\,d\omega. \tag{72}$$

We are led to show that

$$\underline{lim}_{r\to+\infty}f(r)=0,$$

where $f(r)$ is the last integral in (72).

We compute

$$\frac{d}{dr}\int_{S_r}|u|^2dS=\int_{S_r}\frac{d}{dr}|u|^2dS+\int_{C_r}|u|^2\;r\,\vec{n}\cdot\vec{n}\;dl,$$

where the circle C_r is the boundary of S_r.

Then,

$$f(r)=\frac{1}{2}\,\frac{\partial}{\partial r}\int_{S_r}|u|^2dS-r\int_{C_r}|u|^2dl.$$

We find

$$\frac{1}{2}\,\frac{\partial}{\partial r}\int_{S_r}|u|^2dS\geq f(r).$$

But since $u \in L^2(\Omega_R)$,

$$\int_R^{+\infty} dr \left(\int_{S_r} |u|^2 dS \right) < +\infty. \tag{73}$$

If we had $\underline{lim}_{r \to +\infty} \frac{\partial}{\partial r} \int_{S_r} |u|^2 dS = +\infty$, there would exist a positive number a such that, if r_0 is large enough: $\frac{\partial}{\partial r} \int_{S_r} |u|^2 dS \geq a > 0$, for all $r \geq r_0$.

But this is impossible because of (73). Then we conclude that $\underline{lim}_{r \to +\infty} \frac{\partial}{\partial r} \int_{S_r} |u|^2 dS < +\infty$. This implies that $\underline{lim}_{r \to +\infty} f(r) = 0$, which ends the proof. ∎

Let now u be an L^2-solution of (62). By Lemma 2, we have the estimate

$$\int_{\Omega_R \setminus B_R} \left(|\nabla u|^2 + u^2 \right) \ dx < \infty. \tag{74}$$

In polar coordinates, if we denote by S_2^+ the positive hemisphere of S_2,

$$\int_R^{+\infty} dr \ r^2 \int_{S_2^+} \left(\left(\frac{\partial u}{\partial r} \right)^2 + u^2 \right) \ d\omega < \infty. \tag{75}$$

Let us estimate, with $v = ru$, the integral

$$\int_R^{+\infty} dr \int_{S_2^+} \left(\left(\frac{\partial v}{\partial r} \right)^2 + v^2 \right) \ d\omega. \tag{76}$$

We have $\frac{\partial v}{\partial r} = u + r \frac{\partial u}{\partial r}$, so

$$\int_{S_2^+} \left(\left(\frac{\partial v}{\partial r} \right)^2 + v^2 \right) \ d\omega \leq \int_{S_2^+} \left(r^2 \left(\frac{\partial u}{\partial r} \right)^2 + r^2 u^2 + u^2 \right) \ d\omega, \tag{77}$$

and, for R large enough, this last integral is bounded by

$$2r^2 \int_{S_2^+} \left(\left(\frac{\partial u}{\partial r} \right)^2 + u^2 \right) \ d\omega. \tag{78}$$

Then,

$$\int_R^{+\infty} dr \int_{S_2^+} \left(\left(\frac{\partial v}{\partial r} \right)^2 + v^2 \right) \ d\omega < \infty, \tag{79}$$

which implies that

$$\underline{lim}_{r \to \infty} \int_{S_2^+} \left(\left(\frac{\partial u}{\partial r} \right)^2 + u^2 \right) \ d\omega = 0. \tag{80}$$

But this is in contradiction with the result of Lemma 1. This ends the proof of Proposition 3. ∎

5.2. The Decay for a Compact Perturbation

5.2.1. The general case

We can prove that local energy decays toward zero as time increases, for a solution of the problem

$$\begin{aligned} \Box w &= 0, && \text{for } (x,t) \in \Omega_R \times [0, +\infty), \\ w(x,0) &= f(x), && \text{for } x \in \Omega_R, \\ \partial_t w(x,0) &= g(x), && \text{for } x \in \Omega_R, \\ w(x,t) &= 0, && \text{for } (x,t) \in \Gamma_R \times [0, +\infty), \end{aligned} \tag{81}$$

where Ω_R and Γ_R have been defined previously.

In fact, due to Proposition 3, the proof of Theorem 2 applies verbatim and we have the following theorem.

THEOREM 5. *For any initial data $(f,g) \in C_0^\infty$, with support in $\Omega_R \cap B_R$, the local energy $E(w,t,\Omega \cap B_R)$ of the solution of (81) satisfies:*

$$\lim_{t\to+\infty} E(w,t,\Omega_R \cap B_R) = 0. \tag{82}$$

REMARK. The argument of Ralston (see Section 4) also holds in this situation, which allows us to determine precisely the decay of local energy, arbitrarily slow, in this case (see Theorem 4).

5.2.2. The "star-shaped" case

Let us consider the problem (5), where the initial data f and g are compactly supported in $B_R \cap \Omega_R$.

We denote by γ_R the rough part of Γ_R:

$$\gamma_R = \Gamma_R \cap B_R.$$

If $\nu(x)$ is the unit external normal at the point $x \in \Gamma_R$, we suppose that the "star-shaped" condition $\mathcal{SS}$ is achieved on γ_R. We are going to show that the analysis of Lax, Morawetz and Phillips [13] can be extended to our unbounded situation, leading, as in the bounded case, to the exponential decay.

The reasons why this generalization is possible are essentially the following:

(1) A Huygens principle holds in the half-space for the wave equation with Dirichlet boundary conditions;
(2) The orthogonality property between incoming and outgoing spaces $D_\pm$ of [13] is still valid, insuring us that any signal leaving the region $\Omega_R \cap B_R$, will not interact with it in the future.

According to these remarks, the main result of this section is the following theorem.

THEOREM 6. *If the surface Γ_R satisfies the geometric condition $\mathcal{SS}$, then, for any solution w of (5):*

$$E(w,D,t) \leq e^{-\lambda t} \cdot E(w,\Omega_R,0), \tag{83}$$

where the positive constant λ depends only on the geometry of Γ_R.

PROOF. The proof consists of an adaptation of the various steps of [13]. We recall the necessary definitions, and we shall insist only on the differences implied by the unbounded character. ∎

Let us begin with the following elementary result.

LEMMA 6. HUYGENS PRINCIPLE FOR A HALF-SPACE. *Let us denote by $\overline{x} = (x', -x_3)$ the symmetric of $x = (x', x_3)$ with respect to the plane $\Pi_0 \equiv \{x_3 = 0\}$, and by $\overline{f}$ the symmetrized function associated to f by: $\overline{f}(x) = f(\overline{x})$.*

Then we have:

(1) *The solution w_0 of:*

$$\begin{aligned} \Box w_0 &= 0, && \text{for } (x,t) \in \mathbb{R}^3_+ \times [0,+\infty), \\ w_0(x,0) &= f(x), && \text{for } x \in \mathbb{R}^3_+, \\ \partial_t w_0(x,0) &= g(x), && \text{for } x \in \mathbb{R}^3_+, \\ w_0(x,t) &= 0, && \text{for } (x,t) \in \Pi_0 \times [0,+\infty), \end{aligned} \tag{84}$$

with $f,g \in C^\infty(\mathbb{R}^3_+)$, is given by the formula

$$w_0(x,t) = t\left[M_t(g)(x) - M_t(g)(\overline{x})\right] + \partial_t\left[M_t(f)(x) - M_t(f)(\overline{x})\right], \tag{85}$$

where, for any function ψ,

$$M_t(\psi)(x) = \frac{1}{4\pi}\int_{S_2} \psi(x+t\omega)\,d\omega,$$

S_2 *being the two-sphere defined in spherical coordinates by*

$$\{\omega\colon\ (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta),\ \theta\in[0,\pi],\ \phi\in[0,2\pi]\},$$

with $d\omega = \sin\theta d\theta d\phi$.

(2) *If* $V^+(\xi,\tau) = \{(x,t)\in\mathbb{R}^3\times\mathbb{R}\colon |t-\tau| = |x-\xi|\}$ *is the future cone with apex* (ξ,τ), *and if* $\mathcal{S}$ *is the set:*

$$\mathcal{S} = \mathrm{supp}(f)\cup\mathrm{supp}(g)\cup\mathrm{supp}(\overline{f})\cup\mathrm{supp}(\overline{g}),$$

then we have for the support of w_0*:*

$$\mathrm{supp}(w_0)\subset\bigcup_{(\xi,\tau)\in\mathcal{S}} V^+(\xi,\tau).$$

PROOF.

(1) The function $(x,t)\to tM_t(g)(x)+\partial_t M_t(f)(x)$ is the solution of (84) in the whole space $\mathbb{R}^3$, by the classical Kirchoff formula [14], and the boundary condition is satisfied due to the symmetric term.

(2) Symmetry in formula (85) indicates that w_0 is also the solution, of:

$$\begin{aligned} \Box w_0 &= 0, && \text{for } (x,t)\in\mathbb{R}^3\times[0,+\infty),\\ w_0(x,0) &= (f+\overline{f})(x), && \text{for } x\in\mathbb{R}^3,\\ \partial_t w_0(x,0) &= (g+\overline{g})(x), && \text{for } x\in\mathbb{R}^3, \end{aligned} \tag{86}$$

then, Part 2 of Lemma 6 is only the Huygens principle applied to the "thickened" ball containing the "symmetric" supports $\mathcal{S}$. ∎

Let us consider now the essential steps of [13]. We say that a solution $u(x,t)$ is *incoming* (respectively, *outgoing*), with respect to the ball B_R, if $u(x,t) = 0$ for $t<0$ and $x\in B_{\rho-t}\cap\Omega_R$ (respectively, $t>0$ and $x\in B_{\rho+t}\cap\Omega_R$).

Let us denote by $\mathbb{R}^3_+$ the half-space $\mathbb{R}^3_+ = \{x\in\mathbb{R}^3\ :\ x_3>0\}$.

Following [6], we consider the completion $H_D(\mathbb{R}^3_+)$ of $C_0^\infty(\mathbb{R}^3_+)$, for the Dirichlet norm. Then we put

$$\mathcal{H}_0 = H_D(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+).$$

In the same way, we have

$$\mathcal{H} = H_D(\Omega_R)\times L^2(\Omega_R).$$

We also consider the evolution group $U(t)$ (respectively, $U_0(t)$) associated to the operator $\mathcal{A}$ in $\mathcal{H}$ (respectively, $\mathcal{H}_0$).

We denote by D_- (respectively, D_+), the set of initial data for incoming (respectively, outgoing) solutions, and we choose ρ such that γ_R lays inside the region $\Omega_R\cap B_R$. $D_\pm$ are clearly closed subspaces of $\mathcal{H}$.

As in the free-space situation for the bounded obstacle, due to the simple geometric nature of the geometry of $\Gamma_R\setminus\gamma_R$, we have the following orthogonality result.

PROPOSITION 5. *The subspaces* D_+ *and* D_- *are orthogonal.*

Let us first remark that this result certainly does not hold in the general case. In fact, if the rough part of Γ_R is not bounded, a lot of rays, and consequently a lot of energy, may come back in the initial region long in the far future, and some further condition is needed to prevent this come-back.

If we denote by $\mathcal{K}$ the set

$$\mathcal{K} = \{\phi \in \mathcal{H}\colon\ \phi \text{ is orthogonal to } D_+ \text{ and } D_-\},$$

and if $P_\pm$ are the projectors on $D_\pm$, then we have the following proposition.

PROPOSITION 6. *For $t > 0$, the operators*

$$Z(t) = P_+ U(t) P_-,$$

map $\mathcal{H}$ into $\mathcal{K}$, and form a one-parameter semigroup over $\mathcal{K}$.

SKETCH OF THE PROOF. By Proposition 4, the orthogonality between D_+ and D_- insure us that the action of P_+ on a vector $\phi \in D_-^\perp$, leaves it orthogonal to D_-. This shows that $Z(t)$ maps $\mathcal{H}$ into $\mathcal{K}$.

The semigroup property is shown as in [13] ∎

Then, we have the following proposition.

PROPOSITION 7. *If the rough part of the surface Γ_R satisfies the geometric condition $\mathcal{SS}$, then, there exist two positive constants T and λ, such that*

$$||Z(t)|| \le e^{-\lambda t}, \tag{87}$$

for $t > T$.

PROOF. As in [13], it is sufficient to prove that there exists $T > 0$ such that

$$||Z(t)|| \le 1, \tag{88}$$

and to apply the semigroup property of $Z(t)$. ∎

Now (88) is a consequence of Theorem 3: as the surface Γ_R satisfies clearly the geometric condition $\mathcal{SS}$, then, for any solution w of (5):

$$E(w, D, t) \le \frac{C}{t} \cdot E(w, \Omega_R, 0), \tag{89}$$

where the positive constant C depends only on the geometry of Γ_R and the supports of the initial data.

To prove Proposition 4, we observe that the action of the "half-space semigroup" $U_0(t)$ on a vector ϕ in D_+, satisfies the following "propagation" property:

$$U_0(t)\phi = 0, \qquad \text{for } x \in B_{t-\rho} \cap \Omega_R.$$

Then, the proposition will follow from the following lemma and proof.

LEMMA 7. *If $U_0(t)\phi = 0$ in the region $B_{t-\rho} \cap \Omega_R$, then ϕ is orthogonal to D_-.*

PROOF. The proof relies on the preceding Huygens principle (see Lemma 6), and on the following property of unique continuation for the mixed problem.

PROPOSITION 8. *Let $w(x,t)$ a weak solution with finite energy of the wave equation in $\mathbb{R}^3_+$, with Dirichlet boundary condition on the plane $\{x_3 = 0\}$, which vanishes in the region $B_{\rho-|t|} \cap \mathbb{R}^3_+$.*

Then w is identically zero in $\mathbb{R}^3_+$.

The proof of this last property is achieved, as in [13], by considering the associated reduced wave equation, after a Fourier transform in the temporal domain: $w(x,t) \to \hat{w}(x,\mu)$. By standard elliptic regularity up to the boundary, $\hat{w}$ is smooth, and, due to the explicit form of the solution (see Kirchoff formula (85)), it is easy to show that it is an entire function of μ.

The proof is ended by a Liouville type argument.

5.2.3. The "non-trapping" case

In fact, the "star-shaped" condition may be weakened, following the ideas of Morawetz-Ralston-Strauss [11] and Melrose [31].

We are going to show briefly how this extension can be worked out, relying closely on Ralston [32].

We consider the problem (5) with $f = 0$, to simplify the presentation:

$$\begin{aligned} \Box w &= 0, && \text{for } (x,t) \in \Omega_R \times [0,+\infty), \\ w(x,0) &= 0, && \text{for } x \in \Omega_R, \\ \partial_t w(x,0) &= g(x), && \text{for } x \in \Omega_R, \\ w(x,t) &= 0, && \text{for } (x,t) \in \Gamma_R \times [0,+\infty). \end{aligned} \tag{90}$$

Let us consider the following geometrical condition, which implies clearly the preceding $\mathcal{SS}$ condition.

CONDITION 3. NON-TRAPPING. *The domain Ω_R satisfies the "nontrapping" (or $\mathcal{NT}$) condition if, for some $\rho > 0$ such that the rough part γ_R of Γ_R lies in B_ρ, there exists L_ρ such that no geodesic of length L_ρ remains completely within B_ρ.*

If Ω_R satisfies condition $\mathcal{NT}$, the fundamental solution of the problem (5) has a kernel $E(t,x,y)$ which is smooth on the set:

$$(x,y) \in \overline{\Omega}_R \times \overline{\Omega}_R \colon |y| \le \rho, \text{ and } \quad |x| \le t - T(\rho).$$

This is due to the two facts:

(1) It is true for the fundamental solution $\mathcal{E}(t,x,y)$ of (5) in the whole space, as shown in [33],
(2) We have the explicit formula:

$$\forall y \in \Omega_R \colon \; E(t,x,y) = \mathcal{E}(t,x,y) - \mathcal{E}(t,\overline{x},y).$$

If we define the cut-off propagator $E_0(t,x,y) \equiv \theta E(t,x,y)$, where $\theta(t,x)$ is a smooth function on $\mathbb{R}_+ \times \overline{\Omega}_R$ such that:

(1) $\theta(t,x) = h(t)$ near γ_R,
(2) $\theta \equiv 1$ if $t - L_\rho - 1 < |x| < t + \rho - 1$,
(3) $\theta \equiv 0$ if $|x| < t - L_\rho - 2$ and $|x| > t + \rho + 2$.

Then, if we denote by $R(k) = (\Delta_D + k^2)^{-1}$ the resolvent of Δ_D for $\Im m\ k > 0$, it is easy to verify, using the arguments of [34], that , if $\tilde{E}_0(k)$ is the Fourier-Laplace transform of E_0 in the t-variable, there exists positive constants T, α, and β, such that:

(1) if supp $\phi \subset B_\rho \cap \Omega_R$, the function $k \to R(k)\phi|_{B_\rho \cap \Omega_R}$ has an analytic continuation to the domain

$$S_{\alpha,\beta} = \{k \in C \colon \; |\Im m\ k| \le \alpha \, \log |\Re e\ k| - \beta\},$$

(2) $\forall s \ge 0$, and for each ϕ with support in $B_\rho \cap \Omega_R$, and $k \in S_{\alpha,\beta}$, the following estimate holds:

$$||(R(k) - \tilde{E}_0(k))\phi||_s \le \frac{C}{|k|^s} e^{T|\Im m\ k|} ||\phi||_0. \tag{91}$$

As $R(k)$ has a meromorphic continuation in the complex plane (see [34]), we consider the representation of the solution w of (5) by inverse Laplace transform, for any $b > 0$

$$w = -E_0 g - \int_{ib-\infty}^{ib+\infty} e^{-ikt}(R(k) - \tilde{E}_0(k))g \, dk. \tag{92}$$

Then the asymptotic estimate (91) allows us to push down the contour in (92) up to $\Im m\ k = -\alpha < 0$, to obtain the final result in Theorem 7.

THEOREM 7. *If the surface Γ_R satisfies the geometric condition $\mathcal{NT}$, then, for any solution w of (90):*

$$E(w, D, t) \leq e^{-\lambda t} \cdot E(w, \Omega_R, 0), \tag{93}$$

where the positive constant λ depends only on the geometry of Γ_R.

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